In last week's reading and task about problem-solving "moves", I highlighted the later moves—variations, generalizations, extensions, and connections. This week as we wrap up the smOOC, I'd like to share and model the earlier "moves" that have to do with coming up with an answer and composing a solution. I hope that hearing my take on these might help you as you approach writing up the work that you've done on our MiP problem set—if you choose to do so—as well as in your future math work.

When you share with someone else what happened for you as you solved a problem, there are different levels of detail that you can choose to share. Four places along that spectrum are answer, calculation, solution, and narrative.

An **answer** is just that. I don't have much to say about answers. ☺

A **calculation** shows a little more than an answer. It reveals the "shape" of what you did. You might think of it as the residue of the math and thinking that you've done—residue that hasn't particularly been prepared and polished up to be understood by others, but from which they might glean the direction of your thought. I'd note that a lot of the work that's submitted in a math class—e.g. the solving of an equation—is calculation.

A **solution** has polish. It's tight, concise, and sequential. It tells a logical story that begins with the givens of the problem and then leads—step by step—to an answer. It gives reasons. It shows your thinking through illustrations and other representations. It explains how you solved the problem. A solution guides a reader along your thought-path, and reading it would clue them into your strategy so that they could solve a variation of your problem.

For all that a solution contains, it also leaves out a lot—false starts, side adventures, test cases, mistakes, flashes of intuition, and more. Basically, a solution is a rational retelling of a winding, fits-and-starts process. A recounting of what your messy firsthand experience with the problem is a **narrative**. It isn't tidied up. Your whole thought process is shared, or at least large swaths of it are curated for your reader. You don't spare your reader the pains and backtracking that finding an answer cost you. You share not just the right ideas, but what it's like to seek after and eventually *find* the right ideas.

To illustrate these different ways that you can share about what happened for you as you solved a problem, here's a problem, an answer to it, two calculations that lead to the answer, three solutions that describe where the calculations come from, and a narrative of a class of mine tacking the problem. Enjoy!

Problem: How many ways can 12 people be put into pairs?

Answer: 10395

Calculation: $11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 10395$

or $\dfrac{12!}{6! \cdot 2^6} = 10395$

Solution #1:

Pick one of the 12 people; it doesn't matter which one, as everyone will have to get paired with someone eventually. There are 11 ways to partner him with someone. Putting that pair aside, pick one of the remaining 10 people—again, it doesn't matter which one. There are 9 ways to partner her with someone. Continuing in this fashion, we find that there are successively 11, 9, 7, 5, 3, and 1 ways of constructing pairings. Imagining these as stages of a tree of possibilities, we calculate that there must be $11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 10395$ ways of putting 12 people into pairs.

Solution #2:

There are 12! ways of lining the 12 people up in a row:

ABCDEFGHIJKL

Given an order, we can divide up the line into six pairs:

AB CD EF GH IJ KL

Of course, such a process does not lead to 12! different pairings. Any reordering of the pairs does not produce a new pairing:

AB EF GH CD KL IJ

Therefore we are overcounting by a factor of 6!, which is the number of ways that six given pairs can be rearranged. In addition, switching the order in which the two members of a pair stand does not change the pairing:

BA DC FE HG JI LK

Since each of the six pairs can flip-flop positions, this leads to overcounting by 2 six times, or 2^6. The rearrangement of the six pairs among each other and within themselves account for all of the ways that a single pairing up can be expressed in different form; dividing these away, the total number of pairings is $\frac{12!}{6! \cdot 2^6} = 10395$.

Solution #3:

We can think of putting 12 people into pairs as assigning letter cards to each person, with there being two A's, two B's, and so on. If we line up our 12 people in a row, the number of pairings is equal to the number of ways we can arrange the "word" AABBCCDDEEFF underneath them:

people	A	B	C	D	E	F	G	H	I	J	K	L
cards	A	A	B	B	C	C	D	D	E	E	F	F

There are $\frac{12!}{2!2!2!2!2!2!}$ ways of arranging the cards. Of course, if we have two pairs exchange cards—say the "A" group with the "D" group—the pairings will stay the same even though the cards will look different. Since there are 6! ways of exchanging out the cards, we've overcounted by a factor of 6!, and our final answer is $\frac{12!}{6!\cdot 2^6} = 10395$.

Narrative (from my 2007-08 Trigonometry/Analysis class):

At first we just threw out some possible answers (i.e. guesses), including some based upon formulas we'd previously learned, like 144 or 12! or $12+11+10+...+1$. Justin recorded these and much of what follows on the chalkboard. We soon realized that we weren't at all sure whether these answers were right, or if the formulas applied to this particular problem. Putting our formulas aside for the moment (but perhaps hopeful that we'd be able to better apply them later on), we began to try the problem by reasoning from scratch. We quickly decided that after picking a first person (which some were inclined to call "A", others "Grace"), there were 11 ways of pairing her with someone—one way for each of the remaining people. We next tried to determine how many ways there were to partner the next person. We went back and forth between thinking that there were either 10 or 9 ways of doing so. Also, we were inclined to thinking that in the end we should add up all of the possibilities to get our total answer.

Feeling like we weren't making progress with this strategy, Julie pointed out that we could just start listing out the possibilities. The class felt that this would take way too long. Fortunately, Matt soon offered a different approach—thinking about what things would look like once all the pairings had been made. Justin drew a picture of this on the board, something like __ __ __ __ __ __. Matt claimed that there were 11 ways to do each pairing, and so 66 total ways, but was not able to convince the class of his claim with an argument. Meanwhile, Jackson generalized Matt's claim, saying that there would be $\frac{x}{2}(x-1)$ ways of pairing up *x* people.

Not able to find a convincing argument to back up Matt's claim, we again felt stuck. Justin tried to summarize our thought processes so far. Once he had finished, Hannah suggested that we might try out some smaller cases where it would be easier to list out the possibilities. We then started with 4 people and started working out the possible pairings in much the same way as we had at first. Justin observed that there are limitations on what pairings could happen simultaneously—once the first pairing is made, the two remaining people must be paired together. Then Will enumerated the possibilities with 4 people, thus providing a solution as well as an answer of 3. Somewhere during all this, someone remarked that the answer to the case with 2 people was obviously 1 pairing. As we started working on the case with 6 people, class ended for the day.

The next class began with a summary of what happened the previous period; Catherine, who'd been away, suggested an additive pattern similar to one suggested last class. It seemed to match up with our calculations for the first two cases, but broke down when we considered the 6 people case. We soon convinced ourselves that the branching structure that we found early on in our computation (which was really nothing more than an organized list) would continue to give us three further pairing coming after each of the five original pairings (AB, AC, AD, AE, AF); that took care of four of the people, and of course the last two people had to pair up. This yielded a total of 15 possible pairings of 6 people—whereas the additive conjecture of $\frac{5+4+3+2+1}{2}$ gave $\frac{15}{2}$ instead. Not seeing a pattern in our (emphasized-by-colored-chalk) data thus far, we headed onto the case of 8 people.

# of people	# of ways to pair up
2	1
4	3
6	15
8	

Because of our experience of the structure of the 6 people case, the structure of the 8 people case really jumped out at some people after only a little bit of listing, whereas others were less sure. Will quickly conjectured that the answer would be $3 \cdot 5 \cdot 7 = 105$ and pointed out where those numbers were coming from in our listing. After several comments and explanations, everyone was more or less convinced of the pattern and the structure behind it, and we decided we didn't need to do any more listing and could just compute. Justin asked whether anyone was surprised by the fact that 15 had showed up prominently both in the 6 people case and the 8 people case, and several people jumped to point out the nested relationship of the two lists, and therefore the calculations as well. Then we went ahead, seeing how the same structure would hold for every case; thus the answer for the case with 12 people would be $11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 10395$. For good measure, we generalized our solution to the case of 2n people, saying that the number of ways of pairing them up would be $(2n-1) \cdot (2n-3) \cdot ... \cdot 3 \cdot 1$.